

Local times of subdiffusive biased walks on trees

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Summary. Consider a class of null-recurrent randomly biased walks on a super-critical Galton-Watson tree. We obtain the scaling limits of the local times and the quenched local probability for the biased walk in the sub-diffusive case. These results are a consequence of a sharp estimate on the return time, whose analysis is driven by a family of concave recursive equations on trees.

Keywords. Biased random walk on the Galton–Watson tree, local time, concave recursive equations.

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1 Introduction

We are interested in a randomly biased walk $(X_n)_{n \geq 0}$ on a supercritical Galton–Watson tree \mathbb{T} , rooted at \emptyset . For any vertex $x \in \mathbb{T} \setminus \{\emptyset\}$, denote by \overleftarrow{x} its parent. Let $\omega := (\omega(x, \cdot), x \in \mathbb{T})$ be a sequence of vectors such that for each vertex $x \in \mathbb{T}$, $\omega(x, y) \geq 0$ for all $y \in \mathbb{T}$ and $\sum_{y \in \mathbb{T}} \omega(x, y) = 1$. We assume that $\omega(x, y) > 0$ if and only if either $\overleftarrow{x} = y$ or $\overleftarrow{y} = x$.

For the sake of presentation, we add a specific vertex $\overleftarrow{\emptyset}$, considered as the parent of \emptyset . Let us stress that $\overleftarrow{\emptyset} \notin \mathbb{T}$. We define $\omega(\overleftarrow{\emptyset}, \emptyset) := 1$ and modify the vector $\omega(\emptyset, \cdot)$ such that $\omega(\emptyset, \overleftarrow{\emptyset}) > 0$ and $\omega(\emptyset, \overleftarrow{\emptyset}) + \sum_{x: \overleftarrow{x} = \emptyset} \omega(\emptyset, x) = 1$.

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For given ω , $(X_n, n \geq 0)$ is a Markov chain on $\mathbb{T} \cup \{\emptyset\}$ with transition probabilities ω , starting from \emptyset ; i.e. $X_0 = \emptyset$ and

$$P_\omega(X_{n+1} = y \mid X_n = x) = \omega(x, y).$$

For any vertex $x \in \mathbb{T}$, let $(x^{(1)}, \dots, x^{(\nu_x)})$ be its children, where $\nu_x \geq 0$ is the number of children of x . Define $\mathbf{A}(x) := (A(x^{(i)}), 1 \leq i \leq \nu_x)$ by

$$A(x^{(i)}) := \frac{\omega(x, x^{(i)})}{\omega(x, \overleftarrow{x})}, \quad 1 \leq i \leq \nu_x.$$

When all $A(x^{(i)}) = \lambda$ with some positive constant λ , the walk is called λ -biased walk on a Galton-Watson tree and was studied in detail by Lyons, Pemantle and Peres [19, 20]. We mention that several conjectures in [20] still remain open and we refer to Aidekon [3] for an explicit formula on the speed of the λ -biased walk and the references therein for recent developments.

When $A(x^{(i)})$ is also a random variable, the couple (\mathbb{T}, ω) is a marked tree in the sense of Neveu [22], and the biased walk X can be reviewed as a random walk in random environment.

Let us assume that $\mathbf{A}(x), x \in \mathbb{T}$ (including $x = \emptyset$) are i.i.d., and denote the vector $\mathbf{A}(\emptyset)$ by (A_1, \dots, A_ν) for notational convenience. As such, $\nu \equiv \nu_\emptyset$ is the number of children of \emptyset . Denote by \mathbf{P} the law of ω and define $\mathbb{P}(\cdot) := \int P_\omega(\cdot) \mathbf{P}(d\omega)$. In the literature of random walk in random environment, P_ω is referred to the quenched probability whereas \mathbb{P} is the annealed probability.

Define

$$\psi(t) := \log \mathbf{E} \left(\sum_{i=1}^{\nu} A_i^t \right) \in (-\infty, \infty], \quad \forall t \in \mathbb{R}.$$

In particular, $\psi(0) = \log \mathbf{E}(\nu) > 0$ since \mathbb{T} is supercritical. Assume that

$$\sup\{t > 0 : \psi(t) < \infty\} > 1.$$

We shall consider the case when (X_n) is null-recurrent and sub-diffusive. Lyons and Pemantle [18] gave a precise recurrence/transience criterion for randomly biased walks on an arbitrary infinite tree. Their results, together with Menshikov and Petritis [21] and Faraud [10], imply that (X_n) is null recurrent if and only if $\inf_{0 \leq t \leq 1} \psi(t) = 0$ and $\psi'(1) \leq 0$. There are two different situations in the null-recurrent case: Either $\psi'(1) = 0$, then (X_n) has a slow-movement behavior (see [11], and [16] for the localization of X_n and

the study of the local times processes), or $\psi'(1) < 0$, then (X_n) is sub-diffusive (see [15]). Therefore we assume throughout this paper

$$(1.1) \quad \inf_{0 \leq t \leq 1} \psi(t) = 0 \quad \text{and} \quad \psi'(1) < 0.$$

Let us introduce a parameter

$$\kappa := \inf\{t > 1 : \psi(t) = 0\} \in (1, \infty],$$

with $\inf \emptyset := \infty$. We furthermore assume the following conditions:

$$(1.2) \quad \begin{cases} \mathbf{E}\left(\sum_{i=1}^{\nu} A_i\right)^{\kappa} + \mathbf{E}\left(\sum_{i=1}^{\nu} A_i^{\kappa} \log_+ A_i\right) < \infty, & \text{if } 1 < \kappa \leq 2, \\ \mathbf{E}\left(\sum_{i=1}^{\nu} A_i\right)^2 < \infty, & \text{if } \kappa \in (2, \infty], \end{cases}$$

with $\log_+ x := \max(0, \log x)$ for any $x > 0$, and

$$(1.3) \quad \text{the support of } \sum_{i=1}^{\nu} \delta_{\{\log A_i\}} \text{ is non-lattice when } 1 < \kappa \leq 2.$$

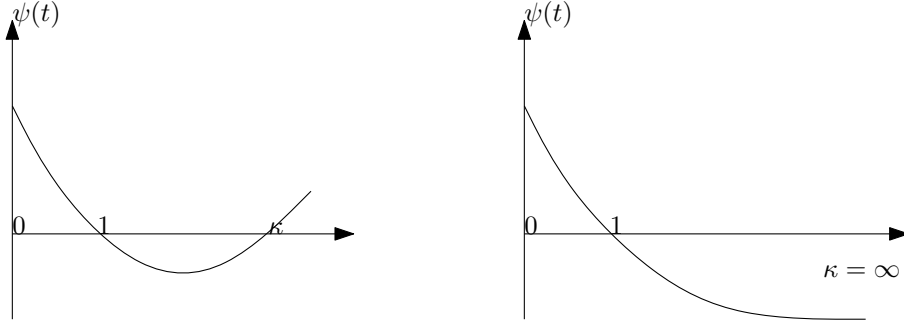


Figure 1: Case $\inf_{0 \leq t \leq 1} \psi(t) = 0$ and $\psi'(1) < 0$: $\kappa \in (1, \infty)$ and $\kappa = \infty$

It was shown in [15] that if ν equals some constant (i.e. \mathbb{T} is a regular tree), then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \log \max_{0 \leq i \leq n} |X_i| = 1 - \max\left(\frac{1}{2}, \frac{1}{\kappa}\right), \quad \mathbb{P}\text{-a.s.}$$

When κ is sufficiently large (say $\kappa \in (5, \infty]$), Faraud [10] proved an invariance principle for the biased walk X , based on the techniques of Peres and Zeitouni [24]; some recent developments cover the whole region $\kappa \in (2, \infty]$ (see Aïdékon and de Raphélis [4] for the convergence to Brownian tree).

The biased walk on a Galton-Watson tree has also attracted many attentions from other directions: In the transient case, Aïdékon [1, 2] dealt with a leafless Galton-Watson tree, whereas Hammond [14] established stable limit laws for the walk on a supercritical Galton-Watson tree with leaves, which can be considered as a counterpart of Ben Arous, Gantert, Fribergh and Hammond [8]. When the tree is sub-critical, Ben Arous and Hammond [9] obtained power laws for the tails of $E_\omega(T_\emptyset^+)$ and the convergence in law of T_\emptyset^+ under a suitable conditional probability, where T_\emptyset^+ denotes the return time to \emptyset :

$$(1.4) \quad T_\emptyset^+ := \inf\{n \geq 1 : X_n = \emptyset\}.$$

In the above-mentioned works [14, 8, 9], the authors explored the link between the biased walk (X_n) and the *trap models* (cf. Ben Arous and Cerny [7]) to get various scaling limits, and an important step is the estimate on the return time to the trap entrance in their models.

We investigate here the return time T_\emptyset^+ in the scope of limit theorems for the local time process of X . It turns out the parameter κ plays a crucial role. Indeed, define (M_n) by

$$(1.5) \quad M_n := \sum_{|x|=n} \prod_{\emptyset < y \leq x} A(y), \quad n \geq 1,$$

where here and in the sequel, $|x|$ is the generation of x in \mathbb{T} and we adopt the partial order: $y < x$ means that y is ancestor of x [we write $y \leq x$ iff either $y < x$ or $y = x$]. Since $\psi(1) = 0$, it is easy to check that (M_n) is a martingale, which in the language of branching random walk is called the additive martingale (cf. Shi [27] further properties on (M_n)). Define

$$\mathbf{P}^*(\bullet) := \mathbf{P}(\bullet \mid \mathbb{T} = \infty),$$

where $\{\mathbb{T} = \infty\}$ denotes the event that the system survives forever. Let $M_\infty := \lim_{n \rightarrow \infty} M_n$ be the almost sure limit of the nonnegative martingale (M_n) . Then under (1.1) and (1.2) [the condition (1.2) is more than necessary to ensure the non-triviality of M_∞], \mathbf{P}^* -a.s. $M_\infty > 0$; If furthermore (1.3) is satisfied (for $1 < \kappa \leq 2$), then

$$(1.6) \quad \mathbf{P}(M_\infty > x) \sim c_M x^{-\kappa},$$

with some positive constant c_M (see Liu [17] Theorems 2.0 and 2.2).

The main estimate on the return time reads as follows. Denote by $f(x) \sim g(x)$ as $x \rightarrow x_0$ if $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$.

Theorem 1.1 Assume (1.1), (1.2) and (1.3). We have that $\mathbf{P}^*(d\omega)$ -a.s.,

$$\frac{1}{\omega(\emptyset, \emptyset) M_\infty} P_\omega(T_\emptyset^+ > n) \sim \begin{cases} c_1 n^{-1/\kappa}, & \text{if } 1 < \kappa < 2, \\ c_2 (n \log n)^{-1/2}, & \text{if } \kappa = 2, \\ c_3 n^{-1/2}, & \text{if } \kappa \in (2, \infty], \end{cases}$$

with

$$\begin{aligned} c_1 &:= \frac{1}{\Gamma(1 - 1/\kappa)} 2^{1/\kappa} (c_M \kappa \mathbb{B}(2 - \kappa, \kappa - 1))^{-1/\kappa}, \\ c_2 &:= (\pi c_M)^{-1/2}, \\ c_3 &:= \left(\frac{2}{\pi} \frac{1 - \mathbf{E}(\sum_{i=1}^\nu A_i^2)}{\mathbf{E}(\sum_{1 \leq i \neq j \leq \nu} A_i A_j)} \right)^{1/2}, \end{aligned}$$

where \mathbb{B} denotes the Beta function and $c_M > 0$ is given in (1.6).

As a consequence, we get the asymptotic behaviors of the local times process:

$$L_n^x := \sum_{i=1}^n 1_{(X_i=x)}, \quad n \geq 1, x \in \mathbb{T}.$$

We shall restrict our attentions to the local times at the root. It was implicitly contained in [15, 6] that for any $\kappa \in (1, \infty]$, \mathbb{P} -almost surely on $\{\mathbb{T} = \infty\}$,

$$L_n^\emptyset = n^{\max(1/\kappa, 1/2) + o(1)}.$$

Based on Theorem 1.1, we can get more precise information on L_n^\emptyset . For any $0 < \alpha < 1$, denote by \mathcal{S}_α a positive stable random variable, independent of (ω, \mathbb{T}) , whose law is determined by the Laplace transform: $\mathbb{E}e^{-\lambda \mathcal{S}_\alpha} = e^{-\lambda^\alpha}, \forall \lambda \geq 0$. It is easy to see, for instance by comparing their Laplace transforms, that $\mathcal{S}_{1/2} \stackrel{(\text{law})}{=} \frac{1}{2\mathcal{N}^2}$, where \mathcal{N} denotes a standard gaussian random variable, centered and with variance 1, independent of (ω, \mathbb{T}) .

Corollary 1.2 Under the same assumptions as in Theorem 1.1, $\mathbf{P}^*(d\omega)$ -a.s., the following convergences in law hold under P_ω :

(i) if $1 < \kappa < 2$, then

$$\frac{L_n^\emptyset}{n^{1/\kappa}} \xrightarrow{(\text{law})} \frac{1}{\omega(\emptyset, \emptyset) M_\infty} \frac{1}{c_1 \Gamma(1 - 1/\kappa)} (\mathcal{S}_{1/\kappa})^{-1/\kappa};$$

(ii) if $\kappa = 2$, then

$$\frac{L_n^\varnothing}{\sqrt{n \log n}} \xrightarrow{(\text{law})} \frac{1}{\omega(\varnothing, \overleftarrow{\varnothing}) M_\infty} \frac{2^{1/2}}{c_2 \pi^{1/2}} |\mathcal{N}|;$$

(iii) if $2 < \kappa \leq \infty$, then

$$\frac{L_n^\varnothing}{\sqrt{n}} \xrightarrow{(\text{law})} \frac{1}{\omega(\varnothing, \overleftarrow{\varnothing}) M_\infty} \frac{2^{1/2}}{c_3 \pi^{1/2}} |\mathcal{N}|.$$

By the classical fluctuation theory on the random walk in the domain of attraction, it is straightforward to deduce from Theorem 1.1 the almost sure limits on L_n^\varnothing : for instance, we have the following law of iterated logarithm:

Corollary 1.3 *Under the same assumptions as in Theorem 1.1, for any $\kappa \in (1, \infty]$, there exists a random variable Υ_κ only depending on (ω, \mathbb{T}) such that $\mathbf{P}^*(\Upsilon_\kappa > 0) = 1$, and on the set $\{\mathbb{T} = \infty\}$,*

$$\limsup_{n \rightarrow \infty} \frac{L_n^\varnothing}{f_\kappa(n)} = \Upsilon_\kappa, \quad \mathbb{P}\text{-almost surely,}$$

where

$$f_\kappa(n) := \begin{cases} n^{1/\kappa} (\log \log n)^{1-1/\kappa}, & \text{if } 1 < \kappa < 2, \\ n^{1/2} (\log n)^{1/2} (\log \log n)^{1/2}, & \text{if } \kappa = 2, \\ n^{1/2} (\log \log n)^{1/2}, & \text{if } \kappa \in (2, \infty]. \end{cases}$$

Combining the estimates on the local times and the reversibility of the biased walk, we obtain the following estimates on the local probability.

Corollary 1.4 *Under the same assumptions as in Theorem 1.1, $\mathbf{P}^*(d\omega)$ -almost surely, for $n \rightarrow \infty$ along the sequence of even numbers, we have*

(i) if $1 < \kappa < 2$, then

$$P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \overleftarrow{\varnothing}) M_\infty} \frac{2 \mathbb{E}(S_\kappa^{-1/\kappa})}{c_1 \kappa \Gamma(1 - 1/\kappa)} n^{1/\kappa - 1};$$

(ii) if $\kappa = 2$, then

$$P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \overleftarrow{\varnothing}) M_\infty} \frac{2}{\pi c_2} n^{-1/2} (\log n)^{1/2};$$

(iii) if $2 < \kappa \leq \infty$, then

$$P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \overleftarrow{\varnothing}) M_\infty} \frac{2}{\pi c_3} n^{-1/2}.$$

2 Outline of the proof

For any $x \in \mathbb{T}$, let $P_{x,\omega}$ be the law of the biased walk X starting from $X_0 := x$. Denote by $E_{x,\omega}$ the expectation under the probability measure $P_{x,\omega}$. In particular, we have $P_{\emptyset,\omega} \equiv P_\omega$ and $E_{\emptyset,\omega} \equiv E_\omega$. Let

$$T_x := \inf\{n \geq 0 : X_n = x\}, \quad x \in \mathbb{T},$$

be the first hitting time of x . Clearly for $n > 2$,

$$\begin{aligned} P_\omega(T_\emptyset^+ > n) &= \sum_{|u|=1} \omega(\emptyset, u) P_{u,\omega}(T_\emptyset > n-1) \\ (2.1) \quad &= \omega(\emptyset, \overleftarrow{\emptyset}) \sum_{|u|=1} A(u) P_{u,\omega}(T_\emptyset > n-1). \end{aligned}$$

By Tauberian theorems, the asymptotic behaviors of $P_{u,\omega}(T_\emptyset > n-1)$, are characterized by that of $E_{u,\omega}(e^{-\lambda T_\emptyset})$ as $\lambda \rightarrow 0$. More generally, we define for any $\lambda > 0$ and $x \in \mathbb{T}$,

$$(2.2) \quad \beta_\lambda(x) := 1 - E_{x,\omega}(e^{-\lambda(1+T_x^\leftarrow)}), \quad x \in \mathbb{T},$$

where as before, \overleftarrow{x} denotes the parent of x . It is easy to see that $\beta_\lambda(\cdot)$ satisfies the following recursive iteration equations:

Fact 2.1 *For any $x \in \mathbb{T}$ and $\lambda > 0$, we have*

$$\beta_\lambda(x) = \frac{(1 - e^{-2\lambda}) + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_\lambda(x^{(i)})}{1 + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_\lambda(x^{(i)})}.$$

We mention that conditioned on $((A(x^{(i)}))_{1 \leq i \leq \nu_x}, \nu_x)$, $(\beta_\lambda(x^{(i)}), 1 \leq i \leq \nu_x)$ are i.i.d. and are distributed as $\beta_\lambda(\emptyset)$.

Proof of Fact 2.1. This fact is an easy application of Markov property. We give the proof for the sake of completeness. For use later, we define for any $n \geq 1, \lambda > 0$ and $x \in \mathbb{T}$ and $|x| \leq n$,

$$(2.3) \quad \beta_{n,\lambda}(x) := 1 - E_{x,\omega}(e^{-\lambda(1+T_x^\leftarrow)} 1_{(\tau_n > T_x^\leftarrow)}),$$

where

$$\tau_n := \inf\{k \geq 0 : |X_k| = n\},$$

denotes the first time when X hits the n -th generation of the tree \mathbb{T} .

Clearly $\beta_{n,\lambda}(x) = 1$ for all $|x| = n$ and for $|x| < n$, we have by the Markov property that

$$\begin{aligned}\beta_{n,\lambda}(x) &= 1 - \left(\sum_{i=1}^{\nu_x} \omega(x, x^{(i)}) e^{-\lambda} E_{\omega, x^{(i)}} e^{-\lambda(1+T_x^{\leftarrow})} 1_{(T_x^{\leftarrow} < \tau_n)} + \omega(x, \overleftarrow{x}) e^{-2\lambda} \right) \\ &= 1 - \left(\sum_{i=1}^{\nu_x} \omega(x, x^{(i)}) (1 - \beta_{n,\lambda}(x^{(i)})(1 - \beta_{n,\lambda}(x)) + \omega(x, \overleftarrow{x}) e^{-2\lambda} \right).\end{aligned}$$

After simplifications, we get that

$$(2.4) \quad \beta_{n,\lambda}(x) = \frac{(1 - e^{-2\lambda}) + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)})}{1 + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)})}, \quad |x| < n.$$

Letting $n \rightarrow \infty$, $\beta_\lambda(x) = \lim_{n \rightarrow \infty} \beta_{n,\lambda}(x)$ and we get Fact 2.1. \square

For brevity, we make a change of variable $\varepsilon = 1 - e^{-2\lambda}$, by defining

$$(2.5) \quad B_\varepsilon(x) := \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{\frac{1}{2} \log 1/(1-\varepsilon)}(x^{(i)}), \quad x \in \mathbb{T}, \quad 0 < \varepsilon < 1,$$

then

$$(2.6) \quad B_\varepsilon(x) = \sum_{i=1}^{\nu_x} A(x^{(i)}) \frac{\varepsilon + B_\varepsilon(x^{(i)})}{1 + B_\varepsilon(x^{(i)})},$$

where as for $\beta_\lambda(x)$, conditioned on $((A(x^{(i)}))_{1 \leq i \leq \nu_x}, \nu_x)$, $(B_\varepsilon(x^{(i)}), 1 \leq i \leq \nu_x)$ are i.i.d. and are distributed as $B_\varepsilon(\emptyset)$.

The main estimate in the proof of Theorem 1.1 will be the following result:

Proposition 2.2 *Assume (1.1), (1.2) and (1.3). As $\varepsilon \rightarrow 0$, the following convergences hold \mathbf{P} -almost surely as well as in $L^p(\mathbf{P})$ for any $1 < p < \min(\kappa, 2)$:*

(i) *if $1 < \kappa < 2$, then*

$$\varepsilon^{-1/\kappa} B_\varepsilon(\emptyset) \rightarrow c_4 M_\infty,$$

where $c_4 := (c_M \kappa \mathbb{B}(2 - \kappa, \kappa - 1))^{-1/\kappa}$, \mathbb{B} denotes the Beta function and $c_M > 0$ is given in (1.6).

(ii) *if $\kappa = 2$, then*

$$\left(\frac{\varepsilon}{\log 1/\varepsilon} \right)^{-1/2} B_\varepsilon(\emptyset) \rightarrow (2 c_M)^{-1/2} M_\infty.$$

(iii) *if $\kappa \in (2, \infty]$, then*

$$\varepsilon^{-1/2} B_\varepsilon(\emptyset) \rightarrow c_5 M_\infty,$$

where $c_5 := \left(\frac{1 - \mathbf{E}(\sum_{i=1}^{\nu} A_i^2)}{\mathbf{E}(\sum_{1 \leq i \neq j \leq \nu} A_i A_j)} \right)^{1/2}$.

Recall that \mathbf{P} -a.s., $\{M_\infty > 0\} = \{\mathbb{T} = \infty\}$. It is straightforward to see that on $\{T = \infty\}^c$, the biased walk X is a Markov chain with finite states, hence $B_\varepsilon(\emptyset) = O(\varepsilon)$ as $\varepsilon \rightarrow 0$.

Let us give the proofs of Theorem 1.1 and Corollaries 1.2, 1.3 and 1.4, by admitting Proposition 2.2:

Proof of Theorem 1.1. By (2.2) and (2.5), we deduce from the usual Abel transform that if $\lambda > 0$ is such that $\varepsilon = 1 - e^{-2\lambda}$, then

$$B_\varepsilon(\emptyset) = (1 - e^{-\lambda}) \sum_{|u|=1} A(u) \sum_{n=0}^{\infty} e^{-\lambda n} P_{u,\omega}(T_\emptyset \geq n).$$

In view of (2.1), $\sum_{|u|=1} A(u) P_{u,\omega}(T_\emptyset \geq n) = P_\omega(T_\emptyset^+ > n) / \omega(\emptyset, \overleftarrow{\emptyset})$. It follows that

$$\sum_{n=0}^{\infty} e^{-\lambda n} P_\omega(T_\emptyset^+ > n) = \omega(\emptyset, \overleftarrow{\emptyset}) \frac{B_\varepsilon(\emptyset)}{1 - e^{-\lambda}},$$

with $\varepsilon = 1 - e^{-2\lambda}$. By the Tauberian theorem ([12], pp. 447, Theorem 5), we immediately obtain Theorem 1.1. \square

Proof of Corollary 1.2. Define for $k \geq 1$,

$$T_\emptyset^{(k)} := \inf\{n > T_\emptyset^{(k-1)} : X_n = \emptyset\},$$

the k -th return to \emptyset (with $T_\emptyset^{(0)} := 0$). Under P_ω , $T_\emptyset^{(k)}$ is the sum of k i.i.d. copies of T_\emptyset^+ , which is in the domain of attraction of a stable law of index $\max(1/\kappa, 1/2)$. We claim that if for some $0 < \alpha < 1$ and some slowly varying function $\ell(n)$,

$$(2.7) \quad P_\omega(T_\emptyset^+ > n) \sim \frac{1}{\Gamma(1 - \alpha)} n^{-\alpha} \ell(n),$$

then under P_ω ,

$$(2.8) \quad n^{-\alpha} \ell(n) L_n^{\emptyset} \xrightarrow{(\text{law})} (\mathcal{S}_\alpha)^{-\alpha},$$

with \mathcal{S}_α the stable law defined in Theorem 1.1. In fact, to get (2.8), we apply [12] (Theorem 2, pp.448) to see that under P_ω ,

$$\frac{T_\emptyset^{(k)}}{(k \ell(k^{1/\alpha}))^{1/\alpha}} \xrightarrow{(\text{law})} \mathcal{S}_\alpha,$$

with \mathcal{S}_α a positive stable variable of index α : $\mathbb{E}e^{-\lambda\mathcal{S}_\alpha} = e^{-\lambda^\alpha}$ for any $\lambda > 0$. Using the fact that $P_\omega(L_n^\varnothing \geq k) = P_\omega(T_\varnothing^{(k)} \leq n)$ for $k \geq 1, n \geq 1$, we easily deduce that for any $z > 0$,

$$P_\omega\left(\frac{L_n^\varnothing}{n^\alpha/\ell(n)} \geq z\right) \rightarrow \mathbb{P}\left(\mathcal{S}_\alpha \leq z^{-1/\alpha}\right),$$

showing (2.8). Clearly, (2.8) implies Corollary 1.2. \square

Proof of Corollary 1.3. It suffices to apply Fristed and Pruitt ([13], Theorem 5) to $T_\varnothing^{(k)}$ (under P_ω). \square

Proof of Corollary 1.4. Under the framework (2.7), we remark that $n^{-\alpha}\ell(n) L_n^\varnothing$ is bounded in $L^p(P_\omega)$ for any $p > 0$. In fact,

$$E_\omega\left(L_n^\varnothing\right)^p \leq \sum_{k=0}^{\infty} p k^{p-1} P_\omega\left(L_n^\varnothing \geq k\right) = \sum_{k=0}^{\infty} p k^{p-1} P_\omega\left(T_\varnothing^{(k)} \leq n\right).$$

Observe that $P_\omega(T_\varnothing^{(k)} \leq n) \leq P_\omega(T_\varnothing^+ \leq n)^k \leq e^{-k P_\omega(T_\varnothing^+ > n)}$. Hence

$$E_\omega\left(L_n^\varnothing\right)^p \leq \sum_{k=0}^{\infty} p k^{p-1} e^{-k P_\omega(T_\varnothing^+ > n)}.$$

Since $\sum_{k=0}^{\infty} p k^{p-1} e^{-kx} \leq C_p x^{-p}$ for all $0 < x \leq 1$ and some constant C_p , we get that $P_\omega(T_\varnothing^+ > n) \times L_n^\varnothing$ is bounded in L^p for any $p > 0$. This together with (2.8) imply that

$$E_\omega(L_n^\varnothing) \sim \mathbb{E}((\mathcal{S}_\alpha)^{-\alpha}) \frac{n^\alpha}{\ell(n)}, \quad n \rightarrow \infty.$$

Under P_ω , the Markov chain X is reversible and it is well-known (see e.g. Saloff-Coste ([26], Lemma 1.3.3 (1), page 323)) that $k \rightarrow P_\omega(X_{2k} = \varnothing)$ is non-increasing. Therefore the Tauberian theorem ([12], formula (5.26), pp.447) yields that

$$P_\omega(X_n = \varnothing) \sim 2\alpha \mathbb{E}((\mathcal{S}_\alpha)^{-\alpha}) \frac{n^{\alpha-1}}{\ell(n)},$$

for $n \rightarrow \infty$ along the sequence of even numbers [the factor 2 comes from the periodicity]. Corollary 1.4 follows. \square

The rest of this paper is devoted to the proof of Proposition 2.2, which will be mainly driven by the recursive equations (2.6). Aldous and Bandyopadhyay [5] pointed out the variety of contexts where the recursive equations have arisen in various models on tree, see also Peres and Pemantle [23] for the studies of a family of concave recursive iterations

using the potential theory. We analyze here the equations (2.6) in the spirit of [15] by establishing some comparison inequalities on the concave iteration.

The key point in the proof of Proposition 2.2 will be the asymptotic behavior of $\mathbf{E}(B_\varepsilon(\emptyset))$. In Section 3, we obtain the lower bound for $\mathbf{E}(B_\varepsilon(\emptyset))$ for all $\kappa \in (1, \infty]$ and get the convergence in law for $\varepsilon^{-1/2}B_\varepsilon(\emptyset)$ for $\kappa \in (2, \infty]$. The upper bound of $\mathbf{E}(B_\varepsilon(\emptyset))$ will be presented in Section 4, where we shall complete the proof of Proposition 2.2 by establishing the almost sure convergence of $\frac{B_\varepsilon(\emptyset)}{\mathbf{E}(B_\varepsilon(\emptyset))}$ to M_∞ .

Throughout this paper, C , C' and C'' (eventually with some subscripts) denote some unimportant constants whose values may vary from one paragraph to another.

3 Concave recursions on trees

Let $0 < \varepsilon < 1$. By (2.6), $B_\varepsilon(\emptyset)$ is a nonnegative solution of the following equation in law:

$$(3.1) \quad B_\varepsilon \stackrel{(\text{law})}{=} \sum_{i=1}^{\nu} A_i \frac{\varepsilon + B_\varepsilon(i)}{1 + B_\varepsilon(i)},$$

where as before, $(A_i, 1 \leq i \leq \nu) \equiv (A(x), |x| = 1)$ and conditioned on $(A_i, 1 \leq i \leq \nu)$, $B_\varepsilon(i)$ are i.i.d., and are distributed as B_ε . We recall that $\mathbf{E}(\sum_{i=1}^{\nu} A_i) = 1$ and $\mathbf{E}(\sum_{i=1}^{\nu} A_i^\kappa) = 1$ if $\kappa < \infty$.

It is easy to get the uniqueness among the nonnegative solutions. Indeed, If B_ε and \tilde{B}_ε are two nonnegative solutions, then in some enlarged probability space, we can find a coupling of $(A_i, 1 \leq i \leq \nu)$, $(B_\varepsilon, B_\varepsilon(i), 1 \leq i \leq \nu)$ and $(\tilde{B}_\varepsilon, \tilde{B}_\varepsilon(i), 1 \leq i \leq \nu)$ such that the equation (3.1) hold a.s. for B_ε and \tilde{B}_ε . Since B_ε is stochastically dominated by $\sum_{i=1}^{\nu} A_i$ hence integrable, we get that $\mathbf{E}|B_\varepsilon - \tilde{B}_\varepsilon| \leq \mathbf{E}|\frac{\varepsilon + B_\varepsilon}{1 + B_\varepsilon} - \frac{\varepsilon + \tilde{B}_\varepsilon}{1 + \tilde{B}_\varepsilon}| \leq (1 - \varepsilon)\mathbf{E}|B_\varepsilon - \tilde{B}_\varepsilon|$ which implies that $B_\varepsilon = \tilde{B}_\varepsilon$ and the claimed uniqueness in law. Therefore we write indistinguishably $B_\varepsilon \equiv B_\varepsilon(\emptyset)$.

This section is devoted to the asymptotic behaviors of $\mathbf{E}(B_\varepsilon)$ as $\varepsilon \rightarrow 0$. Specifically, if $\kappa \in (2, \infty]$ which is the easier case, we shall obtain an exact asymptotic of $\mathbf{E}(B_\varepsilon)$ as $\varepsilon \rightarrow 0$, whereas for $\kappa \in (1, 2]$ we shall get a lower bound, the correspondent upper bound will be proved in Section 4.

First we check that $B_\varepsilon \rightarrow 0$ in $L^1(\mathbf{P})$. Notice that $\mathbf{E}(B_\varepsilon) = \mathbf{E}\frac{\varepsilon + B_\varepsilon}{1 + B_\varepsilon}$ (since $\mathbf{E}(\sum_{i=1}^{\nu} A_i) = 1$), which after simplification gives that

$$(3.2) \quad \mathbf{E}\left(\frac{B_\varepsilon^2}{1 + B_\varepsilon}\right) = \varepsilon \mathbf{E}\left(\frac{1}{1 + B_\varepsilon}\right).$$

Therefore

$$\mathbf{E}(B_\varepsilon) = \mathbf{E} \frac{\varepsilon + B_\varepsilon}{1 + B_\varepsilon} \leq \varepsilon + \mathbf{E} \frac{B_\varepsilon}{1 + B_\varepsilon} \leq \varepsilon + \left(\mathbf{E} \frac{B_\varepsilon^2}{(1 + B_\varepsilon)^2} \right)^{1/2},$$

which in view of (3.2) yield that for any $\kappa \in (1, \infty]$,

$$(3.3) \quad \mathbf{E}(B_\varepsilon) \leq 2\varepsilon^{1/2}, \quad 0 < \varepsilon \leq 1.$$

The above upper bound is sharp (up to a constant) only in the case $\kappa \in (2, \infty]$. To obtain the lower bound on $\mathbf{E}(B_\varepsilon)$, we shall need some inequalities on the concave iteration. Let us adopt the following notation in the rest of this paper:

$$\langle \xi \rangle := \frac{\xi}{\mathbf{E}(\xi)},$$

for any nonnegative random variable ξ with finite mean [as such, $\mathbf{E}\langle \xi \rangle^p = \frac{\mathbf{E}(\xi^p)}{(\mathbf{E}\xi)^p}$].

Lemma 3.1 *Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a convex C^1 -function. For any nonnegative random variable ξ with finite mean and any $0 \leq \varepsilon < 1$, we have*

$$\mathbf{E}\phi\left(\left\langle \frac{\varepsilon + \xi}{1 + \xi} \right\rangle\right) \leq \mathbf{E}\phi(\langle \xi \rangle).$$

Proof: We shall use several times the following inequality in [15], formula (3.3): Let $x_0 \in \mathbb{R}_+$ and let $I \subset \mathbb{R}_+$ be an open interval containing x_0 . Assume that $h : I \times \mathbb{R}_+ \rightarrow (0, \infty)$ is a Borel function such that $\frac{\partial h}{\partial x}$ existe and

- $\mathbf{E}[h(x_0, \xi)] < \infty$ and $\mathbf{E}[\phi(\langle h(x_0, \xi) \rangle)] < \infty$;
- $\mathbf{E}[\sup_{x \in I} (|\frac{\partial h}{\partial x}(x, \xi)| + |\phi'(\langle h(x, \xi) \rangle)| (\frac{|\frac{\partial h}{\partial x}(x, \xi)|}{\mathbf{E}\langle h(x, \xi) \rangle} + \frac{h(x, \xi)}{[\mathbf{E}\langle h(x, \xi) \rangle]^2} |\mathbf{E}(\frac{\partial h}{\partial x}(x, \xi))|))] < \infty$;
- both $y \rightarrow h(x_0, y)$ and $y \rightarrow \frac{\partial}{\partial x} \log h(x, y)|_{x=x_0}$ are monotone on \mathbb{R}_+ .

Then depending on whether $h(x_0, \cdot)$ and $\frac{\partial}{\partial x} \log h(x_0, \cdot)$ have the same monotonicity,

$$(3.4) \quad \frac{d}{dx} \mathbf{E}\phi(\langle h(x, \xi) \rangle) \big|_{x=x_0} \geq 0, \quad \text{or} \quad \leq 0.$$

Applying (3.4) to $h(x, y) := \frac{x+y}{1+y}$, $0 < x < 1$ and $y \geq 0$. For any fixed $x_0 \in (0, 1)$, $h(x_0, \cdot)$ is non-decreasing whereas $\frac{\partial}{\partial x} \log h(x_0, \cdot) = \frac{1}{x_0+}$ is non-increasing. Therefore $x_0 \in (0, 1) \mapsto \mathbf{E}\phi(\langle h(x_0, X) \rangle)$ is non-increasing. It follows that for any $0 < \varepsilon < 1$,

$$\mathbf{E}\phi\left(\left\langle \frac{\varepsilon + \xi}{1 + \xi} \right\rangle\right) \leq \mathbf{E}\phi\left(\left\langle \frac{\xi}{1 + \xi} \right\rangle\right).$$

Now we take $h(x, y) := \frac{y}{1+xy}$ for $x \in (0, 1)$ and $y \geq 0$ in (3.4) and get that $x \in (0, 1) \mapsto \mathbf{E}\phi(\langle \frac{\xi}{1+x\xi} \rangle)$ is non-increasing. Hence $\mathbf{E}\phi(\langle \frac{\xi}{1+\xi} \rangle) \leq \mathbf{E}\phi(\langle \xi \rangle)$ which gives the Lemma. \square

Lemma 3.2 Assume (1.1) and (1.2). For any $p \in (1, 2] \cap (1, \kappa)$, there exists some positive constant $C = C_{p, \kappa}$ such that for any $0 < \varepsilon < 1$,

$$(3.5) \quad \mathbf{E} \left(\langle B_\varepsilon \rangle^p \right) \leq C$$

$$(3.6) \quad \mathbf{E} \left(\left\langle \frac{B_\varepsilon^2}{1 + B_\varepsilon} \right\rangle^p \right) \leq C.$$

Proof of Lemma 3.2: The proof of (3.5) was already given in ([15], Proposition 5.1) in the case that ν equals some integer larger than 2. The same proof can be adopted to the case of random ν and we skip the details.

To prove (3.6), we apply (3.4) to $h(x, y) := \frac{y^2}{x+y}$, $x > 0, y \geq 0$. For any $x_0 > 0$, $h(x_0, \cdot)$ is increasing whereas $y \mapsto \frac{\partial}{\partial x} \log h(x_0, y) = -\frac{1}{x_0+y}$ is also increasing; Hence the function $x \in \mathbb{R}_+ \mapsto \mathbf{E} \left\langle \frac{B_\varepsilon^2}{x+B_\varepsilon} \right\rangle^p$, is non-decreasing on x . Since $\mathbf{E}(B_\varepsilon) \leq 1$, we have

$$\mathbf{E} \left\langle \frac{B_\varepsilon^2}{1+B_\varepsilon} \right\rangle^p \leq \mathbf{E} \left\langle \frac{B_\varepsilon^2}{\frac{1}{\mathbf{E} B_\varepsilon} + B_\varepsilon} \right\rangle^p = \mathbf{E} \left\langle \frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle} \right\rangle^p.$$

Observe that $\mathbf{E} \left(\frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle} \right)^p \leq \mathbf{E}(\langle B_\varepsilon \rangle^p) \leq C$, whereas by Jensen's inequality [the function $x \mapsto \frac{x^2}{1+x}$ is convex], $\mathbf{E}[\frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle}] \geq \frac{1}{2}$ [recalling $\mathbf{E}\langle B_\varepsilon \rangle = 1$], we get that

$$\mathbf{E} \left\langle \frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle} \right\rangle^p = \frac{\mathbf{E} \left(\frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle} \right)^p}{\left(\mathbf{E}[\frac{\langle B_\varepsilon \rangle^2}{1 + \langle B_\varepsilon \rangle}] \right)^p} \leq 2C,$$

yielding (3.6) by eventually choosing a larger constant. \square

To get a lower bound of $\mathbf{E}(B_\varepsilon)$, we shall use the following comparison lemma:

Lemma 3.3 Assume (1.1) and (1.2). For any $a > 0$, let $\phi_a(x) := \frac{x^2}{a+x}$ for any $x \geq 0$. We have

$$\mathbf{E} \phi_a(\langle B_\varepsilon \rangle) \leq \mathbf{E} \phi_a(M_\infty).$$

Proof of Lemma 3.3. Let $a > 0$. It is elementary to check that the function ϕ_a is convex. Moreover, for any $b \geq 0$ and $t > 0$, the function $x \mapsto \phi_a(b + tx)$ is still convex. By Lemma 3.1, we get that for any $b \geq 0$ and $t > 0$,

$$(3.7) \quad \mathbf{E} \phi_a \left(b + t \left\langle \frac{\varepsilon + \xi}{1 + \xi} \right\rangle \right) \leq \mathbf{E} \phi_a(b + t \langle \xi \rangle).$$

Recall (2.3). Choose λ such that $1 - e^{-2\lambda} = \varepsilon$. Define

$$B_n(x) := \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)}), \quad \forall |x| \leq n.$$

Then $B_\varepsilon = B_\varepsilon(\emptyset) = \lim_{n \rightarrow \infty} B_n(\emptyset)$, \mathbf{P} -almost surely. For any $|x| < n$, we deduce from (2.4) that

$$B_n(x) = \sum_{i=1}^{\nu_x} A(x^{(i)}) \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})}.$$

Since $\mathbf{E}(\sum_{i=1}^{\nu_x} A(x^{(i)})) = 1$, we get that

$$\langle B_n(x) \rangle = \sum_{i=1}^{\nu_x} A(x^{(i)}) \left\langle \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right\rangle.$$

Applying (3.7) to $\xi = B_n(x^{(1)})$, $t = A(x^{(1)})$ and $b := 1_{(\nu_x \geq 2)} \sum_{i=2}^{\nu_x} A(x^{(i)}) \left\langle \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right\rangle$ and conditioning on (t, b) , we have that

$$\mathbf{E}\phi_a(\langle B_n(x) \rangle) \leq \mathbf{E}\phi_a\left(1_{(\nu_x \geq 2)} \sum_{i=2}^{\nu_x} A(x^{(i)}) \left\langle \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right\rangle + A(x^{(1)}) \langle B_n(x^{(1)}) \rangle\right).$$

In the right-hand-side of the above inequality, applying (3.7) successively to $B_n(x^{(2)}), \dots, B_n(x^{(\nu_x)})$ with obvious choices of t and b , we get that for any $|x| < n$,

$$\mathbf{E}\phi_a(\langle B_n(x) \rangle) \leq \mathbf{E}\phi_a\left(\sum_{i=1}^{\nu_x} A(x^{(i)}) \langle B_n(x^{(i)}) \rangle\right).$$

Notice by definition $B_n(x) = \sum_{i=1}^{\nu_x} A(x^{(i)})$ for $|x| = n - 1$. By iterating the above inequalities, we get that

$$\mathbf{E}\phi_a(\langle B_n(\emptyset) \rangle) \leq \mathbf{E}\phi_a\left(\sum_{|x|=n} \prod_{\emptyset < y \leq x} A(y)\right) = \mathbf{E}\phi_a(M_n).$$

Lemma 3.3 follows by letting $n \rightarrow \infty$. □

Lemma 3.4 *Assume (1.1), (1.2) and (1.3). We have*

$$(3.8) \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-1/\kappa} \mathbf{E}(B_\varepsilon) \geq c_4, \quad \text{if } 1 < \kappa < 2,$$

$$(3.9) \quad \liminf_{\varepsilon \rightarrow 0} \left(\frac{\log 1/\varepsilon}{\varepsilon}\right)^{1/2} \mathbf{E}(B_\varepsilon) \geq (2c_M)^{-1/2}, \quad \text{if } \kappa = 2,$$

$$(3.10) \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \mathbf{E}(B_\varepsilon) \geq c_5, \quad \text{if } \kappa = 2,$$

where c_4 and c_5 are as in Proposition 2.2 and $c_M > 0$ is given in (1.6).

Proof of Lemma 3.4. If $\kappa \in (2, \infty]$, we remark that $\mathbf{E}(\sum_{i=1}^{\nu} A_i^2) < 1$ and $\mathbf{E}(M_{\infty}^2) = \frac{\mathbf{E}(\sum_{1 \leq i \neq j \leq \nu} A_i A_j)}{1 - \mathbf{E}(\sum_{i=1}^{\nu} A_i^2)}$. It is elementary to check that as $a \rightarrow \infty$,

$$(3.11) \quad \mathbf{E} \frac{M_{\infty}^2}{a + M_{\infty}} \sim \begin{cases} c_M c_{\kappa} a^{1-\kappa}, & \text{if } 1 < \kappa < 2, \\ 2 c_M \frac{\log a}{a}, & \text{if } \kappa = 2, \\ \frac{1}{a} \mathbf{E}(M_{\infty}^2), & \text{if } \kappa \in (2, \infty], \end{cases}$$

where for $1 < \kappa < 2$, $c_{\kappa} := \int_0^{\infty} dy \frac{(2+y)y^{-\kappa+1}}{(1+y)^2} = \kappa \mathbb{B}(2-\kappa, \kappa-1)$.

Recall from (3.2) that $\mathbf{E}(\frac{B_{\varepsilon}^2}{1+B_{\varepsilon}}) = \varepsilon \mathbf{E}(\frac{1}{1+B_{\varepsilon}})$ which can be re-written as

$$(3.12) \quad \mathbf{E} \frac{\langle B_{\varepsilon} \rangle^2}{a + \langle B_{\varepsilon} \rangle} = a \varepsilon \mathbf{E} \left(\frac{1}{1+B_{\varepsilon}} \right) \sim a \varepsilon, \quad \varepsilon \rightarrow 0,$$

where $a \equiv a(\varepsilon) := 1/\mathbf{E}(B_{\varepsilon}) \rightarrow \infty$ by (3.3). By Lemma 3.3,

$$\mathbf{E} \frac{\langle B_{\varepsilon} \rangle^2}{a + \langle B_{\varepsilon} \rangle} \leq \mathbf{E} \frac{M_{\infty}^2}{a + M_{\infty}}.$$

Hence for $a = 1/\mathbf{E}(B_{\varepsilon})$,

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{a \varepsilon} \mathbf{E} \frac{M_{\infty}^2}{a + M_{\infty}} \geq 1,$$

which in view of (3.11) yield the Lemma. \square

We are ready to deal with the asymptotic behaviors of B_{ε} when $\kappa \in (2, \infty]$:

Proposition 3.5 *Assume (1.1) and (1.2). If $\kappa \in (2, \infty]$, then under the probability \mathbf{P} , as $\varepsilon \rightarrow 0$,*

$$\varepsilon^{-1/2} B_{\varepsilon} \xrightarrow{(\text{law})} c_5 M_{\infty},$$

with $c_5 := \left(\frac{1 - \mathbf{E}(\sum_{i=1}^{\nu} A_i^2)}{\mathbf{E}(\sum_{1 \leq i \neq j \leq \nu} A_i A_j)} \right)^{1/2}$ as in Proposition 2.2. Moreover,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \mathbf{E}(B_{\varepsilon}) = c_5.$$

Proof of Proposition 3.5: Based on the boundedness in L^2 of $\varepsilon^{-1/2} B_{\varepsilon}$ (cf. (3.5)), it suffices to prove the convergence in law. Let us first show the tightness of $\frac{B_{\varepsilon}}{\sqrt{\varepsilon}}$ as $\varepsilon \rightarrow 0$. By (3.3) and (3.10),

$$(3.13) \quad c_5 \leq \liminf_{\varepsilon \rightarrow 0} \frac{\mathbf{E}(B_{\varepsilon})}{\varepsilon^{1/2}} \leq \limsup_{\varepsilon \rightarrow 0} \frac{\mathbf{E}(B_{\varepsilon})}{\varepsilon^{1/2}} \leq 2.$$

In particular, under \mathbf{P} , the family of the laws of $(\frac{B_\varepsilon}{\sqrt{\varepsilon}}, \varepsilon \rightarrow 0)$ is tight. For any subsequence $\varepsilon_n \rightarrow 0$ such that $\frac{B_{\varepsilon_n}}{\sqrt{\varepsilon_n}} \xrightarrow{(\text{law})} \xi$, for some nonnegative r.v. ξ . By (3.13), ξ is not degenerate; moreover, we deduce from (3.1) that ξ must satisfy the cascade equation:

$$\xi \stackrel{(\text{law})}{=} \sum_{i=1}^{\nu} A_i \xi_i,$$

where conditioned on (A_i) , ξ_i are i.i.d. copies of ξ . By the uniqueness of the solution (see Liu [17]), $\xi = c M_\infty$ for some positive constant c . We re-write (3.2) as

$$\mathbf{E}\left(\frac{(\frac{B_\varepsilon}{\sqrt{\varepsilon}})^2}{1 + B_\varepsilon}\right) = \mathbf{E}\left(\frac{1}{1 + B_\varepsilon}\right),$$

which by Fatou's lemma along the subsequence $\varepsilon_n \rightarrow 0$, gives that $c^2 \mathbf{E}(M_\infty^2) \leq 1$, i.e. $c \leq (\mathbf{E}(M_\infty^2))^{-1/2} = c_5$. This in view of the lower bound in (3.13) imply that $c = c_5$. Then we have proved that any subsequence of $\frac{B_\varepsilon}{\sqrt{\varepsilon}}$ converges to the same limit $c_5 M_\infty$, which gives the Proposition. \square

4 Proof of Proposition 2.2

To prove Proposition 2.2, it suffices to show the following two statements: As $\varepsilon \rightarrow 0$,

$$(4.1) \quad \mathbf{E}(B_\varepsilon) \sim \begin{cases} c_4 \varepsilon^{1/\kappa}, & \text{if } 1 < \kappa < 2, \\ (2 c_M)^{-1/2} \left(\frac{\varepsilon}{\log 1/\varepsilon}\right)^{1/2}, & \text{if } \kappa = 2, \\ c_5 \varepsilon^{1/2}, & \text{if } \kappa \in (2, \infty]. \end{cases}$$

$$(4.2) \quad \langle B_\varepsilon(\emptyset) \rangle \rightarrow M_\infty, \quad \mathbf{P}\text{-a.s.}$$

The L^p -convergence will follow from (4.2) and the L^p -boundedness of $\langle B_\varepsilon(\emptyset) \rangle$ given in (3.5).

Some preparations first. Using the elementary inequality: $\frac{\varepsilon+x}{1+x} = x + \frac{\varepsilon+x^2}{1+x} \geq x - \frac{x^2}{1+x}$, we deduce from (2.6) that

$$(4.3) \quad \begin{aligned} \langle B_\varepsilon(x) \rangle &= \sum_{y: \overleftarrow{y}=x} A(y) \frac{1}{\mathbf{E}(B_\varepsilon)} \frac{\varepsilon + B_\varepsilon(y)}{1 + B_\varepsilon(y)} \\ &\geq \sum_{y: \overleftarrow{y}=x} A(y) \langle B_\varepsilon(y) \rangle - \sum_{y: \overleftarrow{y}=x} A(y) \Delta(y), \end{aligned}$$

with

$$\Delta(y) := \frac{1}{\mathbf{E}(B_\varepsilon)} \frac{B_\varepsilon(y)^2}{1 + B_\varepsilon(y)},$$

where as before $B_\varepsilon \equiv B_\varepsilon(\emptyset)$, and conditioned on $(A(y), \overleftarrow{y} = x, \nu_x)$, $\Delta(y)$ are i.i.d. copies of $\Delta := \frac{1}{\mathbf{E}(B_\varepsilon)} \frac{B_\varepsilon^2}{1 + B_\varepsilon}$.

To get an upper bound, we dominate $\frac{\varepsilon + B_\varepsilon(y)}{1 + B_\varepsilon(y)}$ by $\varepsilon + B_\varepsilon(y)$, and get that

$$(4.4) \quad \langle B_\varepsilon(x) \rangle \leq \sum_{y: \overleftarrow{y}=x} A(y) \langle B_\varepsilon(y) \rangle + \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \sum_{y: \overleftarrow{y}=x} A(y).$$

By iterating (4.3), we get that for any $m \geq 1$,

$$(4.5) \quad \langle B_\varepsilon \rangle \equiv \langle B_\varepsilon(\emptyset) \rangle \geq \sum_{|x|=m} \prod_{\emptyset < y \leq x} A(y) \langle B_\varepsilon(x) \rangle - \Theta_m,$$

with

$$\Theta_m := \sum_{k=1}^m \sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y) \Delta(x),$$

where conditioned on $(V(x), |x| \leq m)$, $(B_\varepsilon(x), \Delta(x))$ are i.i.d. copies of (B_ε, Δ) . Remark that

$$\mathbf{E}(\Delta) = \frac{1}{\mathbf{E}(B_\varepsilon)} \mathbf{E} \frac{B_\varepsilon^2}{1 + B_\varepsilon} = \frac{1}{\mathbf{E}(B_\varepsilon)} \mathbf{E} \frac{\varepsilon}{1 + B_\varepsilon} \leq \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)}.$$

Consequently,

$$(4.6) \quad \mathbf{E}(\Theta_m) = m \mathbf{E}(\Delta) \leq \frac{\varepsilon m}{\mathbf{E}(B_\varepsilon)}, \quad \forall m \geq 1.$$

Similarly, by iterating (4.4), we get that for any $m \geq 1$,

$$(4.7) \quad \langle B_\varepsilon \rangle \leq \sum_{|x|=m} \prod_{\emptyset < y \leq x} A(y) \langle B_\varepsilon(x) \rangle + \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \sum_{k=1}^m M_k,$$

where as before, $M_k := \sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y)$.

Observe that for any $m \geq 1$, $M_\infty = \sum_{|x|=m} \prod_{\emptyset < y \leq x} A(y) M_\infty^{(x)}$, where conditioned on $(A(x), |x| \leq m)$, $M_\infty^{(x)}$ are i.i.d. copies of M_∞ . Let

$$Y_m := \sum_{|x|=m} \prod_{\emptyset < y \leq x} A(y) (\langle B_\varepsilon(x) \rangle - M_\infty^{(x)}).$$

The following fact is due to Petrov [25], pp. 82, (2.6.20): Let $k \geq 1$ and $1 \leq p \leq 2$. Let ξ_1, \dots, ξ_k be independent random variables such that $\mathbf{E}(|\xi_i|^p) < \infty$ and $\mathbf{E}(\xi_i) = 0$ for all $1 \leq i \leq k$. Then

$$(4.8) \quad \mathbf{E} |\xi_1 + \dots + \xi_k|^p \leq 2 \sum_{i=1}^k \mathbf{E}(|\xi_i|^p).$$

Applying (4.8) to Y_m yields that for any $p \in (1, \kappa) \cap (1, 2]$,

$$\begin{aligned}
\mathbf{E}\left(|Y_m|^p\right) &\leq 2 \mathbf{E}\left(\sum_{|x|=m} \prod_{\emptyset < y \leq x} (A(y))^p\right) \mathbf{E}\left(|\langle B_\varepsilon \rangle - M_\infty|^p\right) \\
(4.9) \qquad &\leq C \left(\mathbf{E} \sum_{i=1}^{\nu} A_i^p\right)^m,
\end{aligned}$$

by using the fact that $\mathbf{E}(|\langle B_\varepsilon \rangle - M_\infty|^p)$ is bounded as $\varepsilon \rightarrow 0$.

To prove (4.1), we only need to get an upper bound of $\mathbf{E}(B_\varepsilon)$ in the case $1 < \kappa \leq 2$. We present a preliminary lemma:

Lemma 4.1 *Assume (1.1) and (1.2). Let $1 < \kappa \leq 2$. For any $0 < \delta < 1$ and $b > 1$, there exists some $\varepsilon_0 \equiv \varepsilon_0(b, \delta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\varepsilon^{1-1/\kappa}(\log 1/\varepsilon)^2 \leq r \leq \varepsilon^{-b}$, we have*

$$\mathbf{P}\left(\langle B_\varepsilon \rangle > r\right) \geq (1 - \delta) \mathbf{P}\left(M_\infty > (1 + \delta)r\right).$$

Proof of Lemma 4.1. Fix a small $\delta > 0$. Let $m \geq 1$. Since $\langle B_\varepsilon \rangle \geq M_\infty + Y_m - \Theta_m$, we get that for any $r > 0$,

$$\mathbf{P}\left(\langle B_\varepsilon \rangle > r\right) \geq \mathbf{P}\left(M_\infty > (1 + \delta)r\right) - \mathbf{P}\left(|Y_m| > \frac{\delta}{2}r\right) - \mathbf{P}\left(\Theta_m > \frac{\delta}{2}r\right).$$

Choose r such that

$$\frac{\delta}{4}r \geq \frac{\varepsilon m}{\mathbf{E}(B_\varepsilon)}.$$

Let $p \in (1, \kappa)$. By (4.6), we have

$$\mathbf{P}\left(\Theta_m > \frac{\delta}{2}r\right) \leq \mathbf{P}\left(\Theta_m - \mathbf{E}(\Theta_m) > \frac{\delta}{4}r\right) \leq \left(\frac{\delta}{4}r\right)^{-p} \mathbf{E}|\Theta_m - \mathbf{E}(\Theta_m)|^p.$$

By the convexity,

$$\begin{aligned}
\mathbf{E}|\Theta_m - \mathbf{E}(\Theta_m)|^p &= \mathbf{E}\left|\sum_{k=1}^m \sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y)(\Delta(x) - \mathbf{E}(\Delta(x)))\right|^p \\
&\leq m^{p-1} \sum_{k=1}^m \mathbf{E}\left|\sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y)(\Delta(x) - \mathbf{E}(\Delta(x)))\right|^p \\
&\leq 2m^{p-1} \sum_{k=1}^m \mathbf{E}\left(\sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y)^p\right) \mathbf{E}|\Delta - \mathbf{E}(\Delta)|^p,
\end{aligned}$$

where the last inequality is a consequence of the use of (4.8). Notice that $\mathbf{E}(\langle \Delta \rangle^p) = \mathbf{E} \left\langle \frac{B^2}{1+B} \right\rangle^p \leq C$ by (3.6). Hence $\mathbf{E}|\Delta - \mathbf{E}(\Delta)|^p \leq C' (\mathbf{E}(\Delta))^p \leq C' \left(\frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \right)^p$. It follows that for any $p \in (1, \kappa)$,

$$(4.10) \quad \mathbf{E}|\Theta_m - \mathbf{E}(\Theta_m)|^p \leq C'' m^{p-1} \left(\frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \right)^p,$$

by using the fact that $\sum_{k=1}^m \mathbf{E}(\sum_{|x|=k} \prod_{\emptyset < y \leq x} A(y)^p) = \sum_{k=1}^m (\mathbf{E} \sum_{i=1}^\nu A_i^p)^k$ is bounded by some constant for all $m \geq 1$. Then for any $p \in (1, \kappa)$, if $\frac{\delta}{4} r \geq \frac{\varepsilon m}{\mathbf{E}(B_\varepsilon)}$, then

$$\mathbf{P}\left(\Theta_m > \frac{\delta}{2} r\right) \leq C (\delta r)^{-p} m^{p-1} \left(\frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \right)^p.$$

For the term Y_m , we use (4.9) and get that

$$\mathbf{P}\left(|Y_m| > \frac{\delta}{2} r\right) \leq C \left(\frac{\delta}{2} r\right)^{-p} (\mathbf{E} \sum_{i=1}^\nu A_i^p)^m.$$

Notice that $\mathbf{E}(\sum_{i=1}^\nu A_i^p) < 1$ for $p \in (1, \kappa)$ ($\kappa \leq 2$). By choosing $m := \lfloor C'_p \log \frac{1}{\varepsilon} \rfloor$ with a sufficiently large constant C'_p , we get that for any $p \in (1, \kappa)$ and for all r such that $\frac{\delta}{4} r \geq C'_p \frac{\varepsilon \log \frac{1}{\varepsilon}}{\mathbf{E}(B_\varepsilon)}$,

$$\begin{aligned} \mathbf{P}\left(\langle B_\varepsilon \rangle > r\right) &\geq \mathbf{P}\left(M_\infty > (1 + \delta)r\right) - (\delta r)^{-p} C_p \left(\log \frac{1}{\varepsilon}\right)^{p-1} \left(\frac{\varepsilon}{\mathbf{E}(B_\varepsilon)}\right)^p \\ &\geq \mathbf{P}\left(M_\infty > (1 + \delta)r\right) - (\delta r)^{-p} \varepsilon^{p-p/\kappa+o(1)}, \end{aligned}$$

where the last inequality follows from the lower bound of $\mathbf{E}(B_\varepsilon)$ in Lemma 3.4.

By the lower tail of M_∞ (see Liu [17], Theorem 2.2), there exists some positive constant C such that

$$\mathbf{P}\left(M_\infty > (1 + \delta)r\right) \geq C r^{-\kappa}, \quad \forall r \geq 1, 0 < \delta < 1,$$

which easily yields Lemma 4.1 since p can be chosen as close to κ as possible. \square

We now have all ingredients to give the proof of (4.1).

Proof of (4.1): By using Proposition 3.5, it remains to show the upper bound of $\mathbf{E}(B_\varepsilon)$ when $1 < \kappa \leq 2$.

Recall from (3.12) that

$$(4.11) \quad \mathbf{E} \frac{\langle B_\varepsilon \rangle^2}{\frac{1}{\mathbf{E}(B_\varepsilon)} + \langle B_\varepsilon \rangle} \sim \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)}, \quad \varepsilon \rightarrow 0.$$

Let $b = 1/(\kappa - 1)$ and $0 < \delta < 1$. Applying Lemma 4.1 to $\delta \leq r \leq \varepsilon^{-b}$, we have by the integration by parts that (with $a := \frac{1}{\mathbf{E}(B_\varepsilon)}$)

$$\begin{aligned}
\mathbf{E} \frac{\langle B_\varepsilon \rangle^2}{a + \langle B_\varepsilon \rangle} &\geq \int_\delta^{\varepsilon^{-a}} \frac{r(2a+r)}{(a+r)^2} \mathbf{P}(\langle B_\varepsilon \rangle > r) dr \\
&\geq (1-\delta) \int_\delta^{\varepsilon^{-a}} \frac{r(2a+r)}{(a+r)^2} \mathbf{P}(M_\infty > (1+\delta)r) dr \\
(4.12) \quad &= (1-\delta) \mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} 1_{(\delta \leq M_\infty \leq \varepsilon^{-a})} \right].
\end{aligned}$$

Observe that $\mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} 1_{(M_\infty \leq \delta)} \right] \leq \frac{\delta^2}{a}$, and

$$\mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} 1_{(M_\infty > \varepsilon^{-b})} \right] \leq C \int_{\varepsilon^{-b}}^\infty \frac{r^2}{a+r} r^{-\kappa-1} dr \leq \frac{C}{\kappa-1} \varepsilon^{b(\kappa-1)}.$$

Therefore

$$\mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} 1_{(\delta \leq M_\infty \leq \varepsilon^{-a})} \right] \geq \mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} \right] - \frac{\delta^2}{a} - \frac{C}{\kappa-1} \varepsilon^{b(\kappa-1)},$$

with $a := \frac{1}{\mathbf{E}(B_\varepsilon)}$. In view of (3.11), we get that for any $1 < \kappa \leq 2$, as $\varepsilon \rightarrow 0$,

$$\mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} 1_{(\delta \leq M_\infty \leq \varepsilon^{-b})} \right] \sim \mathbf{E} \left[\frac{M_\infty^2}{a + M_\infty} \right] \sim \begin{cases} c_M c_\kappa a^{1-\kappa}, & \text{if } 1 < \kappa < 2, \\ 2 c_M \frac{\log a}{a}, & \text{if } \kappa = 2, \end{cases}$$

which together with (4.11) and (4.12) yield the desired upper bound for $\mathbf{E}(B_\varepsilon)$. Thus we get (4.1). \square

Now we are ready to prove (4.2):

Proof of (4.2): Assembling (4.5) and (4.7), we get that for any $m \geq 1$,

$$(4.13) \quad |\langle B_\varepsilon(\emptyset) \rangle - M_\infty| \leq \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \sum_{k=1}^m M_k + |Y_m| + \Theta_m.$$

Denote by $\|\cdot\|_p$ the L^p -norm with respect to \mathbf{P} . It follows that for any $p \in (1, \kappa) \cap (1, 2]$ and $m \geq 1$,

$$\begin{aligned}
\|\langle B_\varepsilon(\emptyset) \rangle - M_\infty\|_p &\leq \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} \sum_{k=1}^m \|M_k\|_p + \|Y_m\|_p + \|\Theta_m\|_p \\
&\leq C \frac{\varepsilon}{\mathbf{E}(B_\varepsilon)} m + C \left(\mathbf{E} \sum_{i=1}^\nu A_i^p \right)^{m/p}
\end{aligned}$$

by using (4.9), (4.6), (4.10) and the fact that $\sup_{k \geq 1} \|M_k\|_p < \infty$. Since $\mathbf{E} \sum_{i=1}^\nu A_i^p < 1$, we choose $m = \lfloor C' \log 1/\varepsilon \rfloor$ with a sufficiently large constant C' and use the lower bound of $\mathbf{E}(B_\varepsilon)$ in Lemma 3.4. This leads to that for any $p \in (1, \kappa) \cap (1, 2]$

$$\|\langle B_\varepsilon(\emptyset) \rangle - M_\infty\|_p \leq \varepsilon^{\frac{1}{2}+o(1)}.$$

Let $\varepsilon_n := n^{-2}$. The convergence part of the Borel-Cantelli lemma yields that as $n \rightarrow \infty$,

$$\langle B_{\varepsilon_n}(\emptyset) \rangle \rightarrow M_\infty, \quad \mathbf{P}\text{-a.s.}$$

Observe that $\varepsilon \rightarrow B_\varepsilon$ is non-increasing, hence for any $\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$, $\langle B_{\varepsilon_{n-1}}(\emptyset) \rangle \frac{\mathbf{E}(B_{\varepsilon_n})}{\mathbf{E}(B_{\varepsilon_{n-1}})} \leq \langle B_\varepsilon(\emptyset) \rangle \leq \langle B_{\varepsilon_n}(\emptyset) \rangle \frac{\mathbf{E}(B_{\varepsilon_{n-1}})}{\mathbf{E}(B_{\varepsilon_n})}$, which readily yield (4.2). This completes the proof of Proposition 2.2. \square

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